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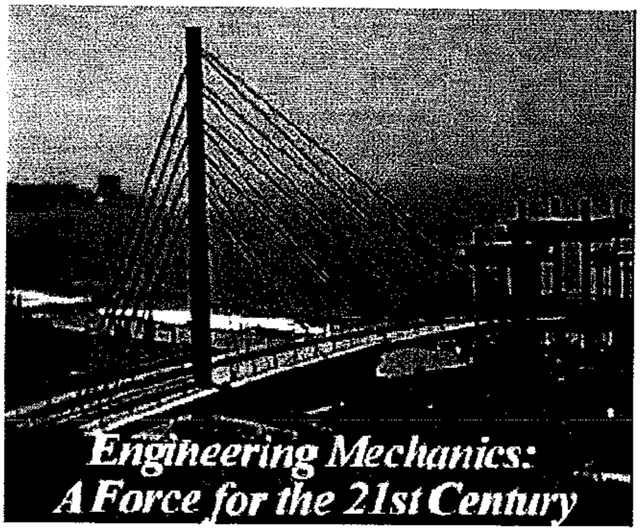

A New Non-Gaussian Closure Method for the PDF Solution of Non-Linear Random Vibrations

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Abstract

The probability density function (PDF) of the response of non-linear stochastic system excited by white noise is assumed to be an exponential function of polynomials. Weight functions are used to make the FPK equation satisfied in weak senses with the assumed PDF. Example is given to show the application of the method to non-linear systems. The PDFs obtained with the proposed method and conventional Gaussian closure method are compared with the exact ones. Numerical results showed that the PDF obtained with the proposed method are much close to the exact PDF. In some cases, even exact results can be obtained with the proposed method.

Introduction

There are various methods for the approximate PDF solutions of non-linear systems with random excitations. Among them, the simplest and also the most popularly used method is the equivalent linearization (Caughey, 1959), which can be taken as a first order approximation. In the special case of Gaussian white noise excitations, this procedure is equivalent to another method, called Gaussian closure (Iyengar and Dash, 1978). However, this method is considered unsuitable when the system is highly non-linear, or when multiplicative random excitations are present, because in either case the probability distribution of the system response is usually far from being Gaussian. To improve the accuracy of an approximate solution, a non-Gaussian closure method was used (Assaf and Zirkie, 1976). Some other methods were also used for the PDF solutions (Roberts and Spanos, 1986, Lin and Cai, 1988, and Er, 1998).

In this paper, a new closure method is developed with which the probability

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density function (PDF) of the response of non-linear stochastic system is assumed to be an exponential function of polynomials. Weight functions are used to make the FPK equation satisfied in the weak sense of integration with the assumed PDF. The problem of evaluating the parameters in the approximate PDF finally results in solving some simultaneous quadratic algebraic equations. Example is given to show the effectiveness of the method.

Problem Formulation

The non-linear random vibration can be described by the following random differential equations:

$$\frac{d}{dt}X_i = f_i(\mathbf{X}) + g_{ij}(\mathbf{X})\xi_j(t) \qquad i = 1, 2, \ldots, n_x; j = 1, 2, \ldots, m \tag{1}$$

where X_i are components of the system response vector $\mathbf{X}$, and $\xi_j(t)$ are random excitations. Moreover, each X_i is assumed to be distributed on the entire range $(-\infty, +\infty)$, unless stated otherwise. Function f_i and g_{ij} are generally non-linear; however, their functional forms are assumed to be deterministic. When the excitations $\xi_j(t)$ are Gaussian white noises with zero mean and cross-correlation

$$E[\xi_j(t)\xi_k(t+\tau)] = 2\pi K_{jk}\delta(\tau) \tag{2}$$

where $\delta(\tau)$ is Dirac function and K_{jk} are constants, representing the cross-spectral density of ξ_j and ξ_k, the system response $\mathbf{X}$ is a Markov vector and the probability density of the stationary responses is governed by the following Fokker-Planck and Kolmogorov (FPK) equation

$$\frac{\partial}{\partial x_j}(A_j p) - \frac{1}{2}\frac{\partial^2}{\partial x_i \partial x_j}(B_{ij}p) = 0 \tag{3}$$

where $p = p(\mathbf{x})$, x_j are state variables, and the first and second derivative moments A_i and B_{ij} can be derived from (1) as follows:

$$A_i(\mathbf{x}) = f_i(\mathbf{x}) + \pi K_{ls}g_{rs}(\mathbf{x})\frac{\partial}{\partial x_r}g_{il}(\mathbf{x}) \tag{4}$$

$$B_{ij}(\mathbf{x}) = 2\pi K_{ls}g_{il}(\mathbf{x})g_{js}(\mathbf{x}) \tag{5}$$

where $\mathbf{x} = \{x_1, x_2, \cdots, x_{n_x}\}^T$ is state vector and Wong and Zakai's modification (Wong and Zakai, 1965) on physical system is considered in equation (4).

The New Non-Gaussian Closure Method

The PDF $\overline{p}(\mathbf{x}; \mathbf{a})$ for the approximate solution of the reduced FPK equation is assumed to be of the form

$$\overline{p}(\mathbf{x}; \mathbf{a}) = c\exp^{Q_n(\mathbf{x};\mathbf{a})} \tag{6}$$

in which $\mathbf{a}$ is unknown parameter vector, $\mathbf{a} = \{a_1, a_2, \cdots, a_{N_p}\}$ and N_p is the total number of unknown parameters.

It can be shown that, if the left-hand sides of the following equations are integrable and $\overline{p}(\mathbf{x};\mathbf{a})$ exists for $\mathbf{x} \in \Re^{n_x}$, (3) can also be satisfied in the weak sense of integration:

$$\int_{\Re^n} \{A_j \frac{\partial Q_n}{\partial x_j} - \frac{1}{2}(\frac{\partial B_{ij}}{\partial x_j}\frac{\partial Q_n}{\partial x_i} + \frac{\partial B_{ij}}{\partial x_i}\frac{\partial Q_n}{\partial x_j} + B_{ij}\frac{\partial^2 Q_n}{\partial x_i \partial x_j} + B_{ij}\frac{\partial Q_n}{\partial x_i}\frac{\partial Q_n}{\partial x_j})$$

$$+\frac{\partial A_j}{\partial x_j} - \frac{1}{2}\frac{\partial^2 B_{ij}}{\partial x_i \partial x_j}\} h_k(\mathbf{x}) d\mathbf{x} = 0, \qquad k = 1, 2, \cdots, N_p \tag{7}$$

where $h_k(\mathbf{x}), k = 1, 2, \cdots, N_p$, are basic functions which span $\Re^{N_p}$.

By selecting $h_k(\mathbf{x})$ as $x_1^{k_1} x_2^{k_2} \cdots x_{n_x}^{k_{n_x}} f_N(\mathbf{x})$, being $k_1, k_2, \cdots, k_{n_x} = 0, 1, 2, \cdots, N_p$ and $k = k_1 + k_2 + \cdots + k_{n_x}$, (7) result in N_p simultaneous algebraic equations as following:

$$F_k(a_1, a_2, \cdots, a_{N_p}) = 0 \qquad k = 1, 2, \cdots, N_p \tag{8}$$

Numerical experience showed that an effective choice for $f_N(\mathbf{x})$ is the PDF solution of (1) from Gaussian closure. Because of the particular choice of $f_N(\mathbf{x})$, the integartion in Eqs. (7) can be easily evaluated by taking into account for the relationships between higher and lower order moments of normal stochastic processes.

Example

Consider the following Duffing oscillator with additional Gaussian white noise excitation:

$$\ddot{X} + \alpha \dot{X} + (X + \varepsilon_1 X^3 + \varepsilon_2 X^5) = \sqrt{\alpha} W(t) \tag{9}$$

where ε_1 and ε_2 are parameters representing the degree of non-linearity and $W(t)$ is a Gaussian white noise with a correlation function $E[W(t)W(t+\tau)] = 2\delta(\tau)$. (9) has been normalized so that when $\varepsilon_1 = 0$ and $\varepsilon_2 = 0$ the stationary mean square displacement and velocity are both equal to unit.

As it is well known that the stationary PDF of the $X = X_1$ and $\dot{X} = X_2$ is

$$p(x_1, x_2) = C\exp(-\frac{1}{2}x_1^2 - \frac{\varepsilon_1}{4}x_1^4 - \frac{\varepsilon_2}{6}x_1^6 - \frac{1}{2}x_2^2) \tag{10}$$

where C is the normalization constant.

Guassian closure method can give desirable PDF solution if both ε_1 and ε_2 are small. But it is not suitable for highly non-linear problem.

Numerical results showed that the solution obtained with the proposed method is same as the exact result (10) with $\varepsilon_2 = 0$ if $n = 4$, no matter what value the ε_1 is.

Numerical experience also showed that the approximate PDFs obtained with the proposed method are very close to exact solutions for both weakly and highly non-linear systems if $n > 2$. In the case of $\varepsilon_1 = \varepsilon_2 = 5$, the logarithmic PDFs are shown and compared in Fig. 1.

Conclusions

(1) The method is not limited to the degree of non-linearity of systems. For highly non-linear problem, the results from the new method with $n = 4$ are much improved comparing to those obtained with Gaussian closure method. In some cases, even some exact results can be obtained. (2) The results evaluated with the proposed method by using 4th degree polynomials agree well with exact ones, specially in the tails of the PDF. (3) The proposed method can not be limited to the problems with Gaussian excitations or white noises. If the PDF of responses of a system can be described with generalized FPK equation, the method is also valid. The solution procedure is similar.

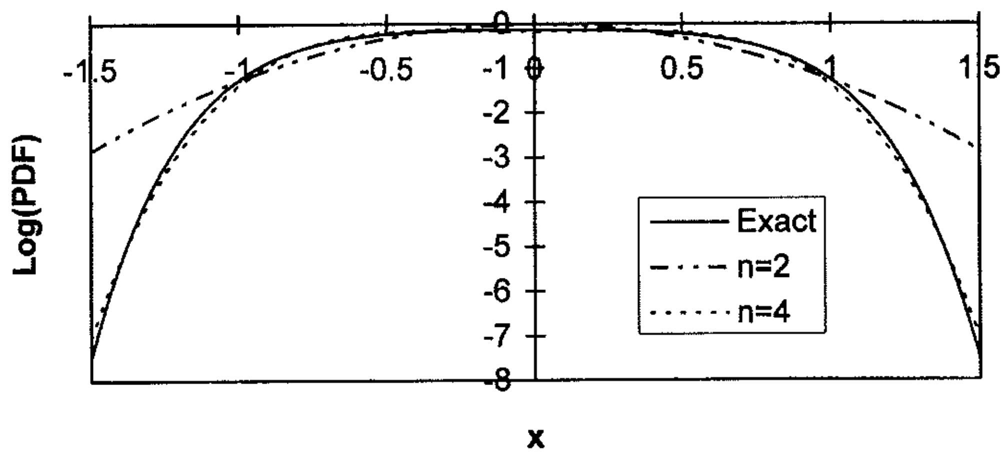

Figure 1. Comparisons of Logarithmic PDFs

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