Annalen der Physik

Founded in 1790

WILEY-VCH

www.ann-phys.org

REPRINT

Ann. Phys. (Berlin) **523**, No. 3, 247 – 258 (2011) / **DOI** 10.1002/andp.201010465

Methodology for the solutions of some reduced Fokker-Planck equations in high dimensions

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Received 14 October 2009, revised 22 October 2010, accepted 8 December 2010 by G. Röpcke
Published online 10 January 2011

Key words Nonlinear stochastic dynamics, large-scale systems, Fokker-Planck equation, state space split.

In this paper, a new methodology is formulated for solving the reduced Fokker-Planck (FP) equations in high dimensions based on the idea that the state space of large-scale nonlinear stochastic dynamic system is split into two subspaces. The FP equation relevant to the nonlinear stochastic dynamic system is then integrated over one of the subspaces. The FP equation for the joint probability density function of the state variables in another subspace is formulated with some techniques. Therefore, the FP equation in high-dimensional state space is reduced to some FP equations in low-dimensional state spaces, which are solvable with exponential polynomial closure method. Numerical results are presented and compared with the results from Monte Carlo simulation and those from equivalent linearization to show the effectiveness of the presented solution procedure. It attempts to provide an analytical tool for the probabilistic solutions of the nonlinear stochastic dynamics systems arising from statistical mechanics and other areas of science and engineering.

1 Introduction

Many practical problems in statistical mechanics and other areas of science and engineering are described as multi-degree-of-freedom (MDOF) or high-dimensional nonlinear stochastic dynamic (NSD) systems [1–13]. Hence, the investigation on the probabilistic solutions of MDOF systems attracted many researchers in the last decades. However, obtaining the probabilistic solutions of MDOF NSD systems has been a challenge for almost a century since the explanation on the motion of molecules by Einstein in 1905 and the formulation of Fokker-Planck (FP) equation thereafter [14–17].

It is generally difficult to obtain the exact solutions of FP equations if the system is nonlinear. Only under restrictive conditions, some exact solutions are obtainable for one, two, or few dimensional systems [1, 18, 19]. As well known, practical systems are normally multi-dimensional or high-dimensional. Therefore, some methods were proposed for the approximate solutions of NSD systems, FP equations, or reduced FP equations. The most frequently used approximation method is the equivalent linearization (EQL) method [2, 9]. The advantage of EQL method is that it can be used for analyzing large-scale NSD systems, but it is considered unsuitable when the system is not weakly nonlinear or multiplicative random excitations are present because in either case the probability distribution of the system responses is usually far from being Gaussian. It is easy to show with the available analytical probabilistic solutions of NSD systems, such as the duffing oscillator excited by Gaussian white noise, that the tails of the probabilistic solution are far from being Gaussian even if the system nonlinearity is considered as slight.

To improve the accuracy of an approximate solution, a non-Gaussian closure method was used [20]. With this method, the probability density function (PDF) of the system responses is approximated with Gram-Charlier series. As well known, this series is not consistent with probability theory, e.g. negative probabilities may result and it is difficult to extend for high-dimensional systems. The principle of maximum entropy was attempted for the approximate probabilistic solutions of NSD systems [21,22], but highly

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nonlinear algebraic equations must be solved in the determination of the unknown parameters and it is also difficult to extend for higher-dimensional or MDOF systems. Stochastic average method is another method for the PDF solutions of the response amplitudes of NSD systems, but it is only suitable for the slightly damped systems with weak excitations and few dimensions [23]. Perturbation method was investigated for the approximate solutions of FP equations [24–26]. It is known that perturbation method is based on the conditions that the solutions of some given systems are known, which is suitable for the nonlinear systems with small parameters and not suitable for high-dimensional or MDOF systems. Monte Carlo simulation (MCS) is versatile [27,28], but the amount of computation with it is usually unacceptable for obtaining the PDF solutions, especially for small probability problems. Exponential polynomial closure (EPC) method was proposed which is suitable for analyzing MDOF or multidimensional systems without being limited by multiplicative excitations and the level of system nonlinearity [29,30]. However the EPC method is only suitable for the systems with few degrees of freedom because the number of unknown parameters in the joint PDF of state variables can increase very fast as the number of state variables increases.

From the above discussion, it is seen that there is no effective analytical method available for obtaining the accurate probabilistic solutions of large-scale NSD systems though practical problems are frequently modeled as large-scale NSD systems. In this paper, a new methodology named state-space-split (SSS) is presented for obtaining the probabilistic solutions of large-scale NSD systems. With the idea of this method, the problem of solving the FP equation in high-dimensional state space becomes the problem of solving some FP equations in low-dimension state spaces. Thereafter, the EPC method can be employed to solve the FP equations in the low-dimensional state spaces. Numerical examples are presented and the numerical results obtained with the presented methodology are compared with those from MCS and EQL to show the effectiveness of the presented solution procedure.

2 Problem formulation

In the following discussion, the summation convention applies unless stated otherwise. The random state variable or vector is denoted with capital letter and the corresponding deterministic state variable or vector is denoted with the same letter in low case.

A lot of problems in science and engineering can be described with the following stochastic dynamic system:

$$\frac{d}{dt}X_i = b_i(\mathbf{X}) + g_{ij}(\mathbf{X})W_j(t) \qquad i = 1, 2, \ldots, n_{\mathbf{x}} \tag{1}$$

in Stratonovich form, where $\mathbf{X} \in \Re^{n_{\mathbf{x}}}$; X_i, $(i = 1, 2, \ldots, n_{\mathbf{x}})$, are components of the state vector process $\mathbf{X}$; $b_i(\mathbf{X}) : \Re^{n_{\mathbf{x}}} \to \Re$; and $g_{ij}(\mathbf{X}) : \Re^{n_{\mathbf{x}}} \to \Re$. Function $b_i(\mathbf{X})$ and $g_{ij}(\mathbf{X})$ are generally nonlinear, and their functional forms are assumed to be deterministic. Here it is assumed that the nonlinearity of the functions $b_i(\mathbf{X})$ and $g_{ij}(\mathbf{X})$ are of polynomial type. The excitations $W_j(t)$ are white noises with zero mean and cross-correlation

$$E[W_j(t)W_k(t+\tau)] = S_{jk}\delta(\tau) \tag{2}$$

where $\delta(\tau)$ is Dirac function and S_{jk} are constants, representing the cross-spectral density of W_j and W_k. Equation (1) can also be expressed in Ito's form as

$$\frac{d}{dt}X_i = f_i(\mathbf{X}) + g_{ij}(\mathbf{X})W_j(t) \qquad i = 1, 2, \ldots, n_{\mathbf{x}} \tag{3}$$

where

$$f_i(\mathbf{X}) = b_i(\mathbf{X}) + \frac{1}{2}\frac{\partial \mathbf{g}_{ij}(\mathbf{X})}{\partial X_m} g_{mj}(\mathbf{X}). \tag{4}$$

The state vector process $\mathbf{X}$ is Markovian and the PDF $p(\mathbf{x},t)$ of the Markov vector is governed by FP equation. Without loss of generality, consider the case when the white noises are Gaussian. In this case, the PDF $p(\mathbf{x},t)$ of the Markov vector is governed by the following FP equation [4]:

$$\frac{\partial p(\mathbf{x},t)}{\partial t} + \frac{\partial}{\partial x_j}\left[f_j(\mathbf{x})p(\mathbf{x},t)\right] - \frac{1}{2}\frac{\partial^2}{\partial x_i \partial x_j}\left[G_{ij}(\mathbf{x})p(\mathbf{x},t)\right] = 0 \tag{5}$$

where $\mathbf{x}$ is the deterministic state vector and $\mathbf{x} \in \Re^{n_\mathbf{x}}$, and

$$G_{ij}(\mathbf{x}) = S_{ls} g_{il}(\mathbf{x}) g_{js}(\mathbf{x}). \tag{6}$$

In stationary state, $\partial p(\mathbf{x},t)/\partial t = 0$. Then

$$\frac{\partial}{\partial x_j}\left[f_j(\mathbf{x})p(\mathbf{x})\right] - \frac{1}{2}\frac{\partial^2}{\partial x_i \partial x_j}\left[G_{ij}(\mathbf{x})p(\mathbf{x})\right] = 0 \tag{7}$$

which is the reduced FP equation.

It is assumed that the solution $p(\mathbf{x},t)$ in Eq. (5) or the solution $p(\mathbf{x})$ in (7) fulfills the following conditions:

$$\lim_{x_i \to \infty} f_j(\mathbf{x})p(\mathbf{x},t) = 0 \quad \text{and} \quad \lim_{x_i \to \infty} \frac{\partial [G_{ij}(\mathbf{x})p(\mathbf{x},t)]}{\partial x_i} = 0 \qquad i,j = 1,2,\cdots,n_\mathbf{x} \tag{8}$$

which can usually be fulfilled by system responses.

3 State space split method

Separate the state vector $\mathbf{X}$ into two parts $\mathbf{X}_1 \in \Re^{n_{\mathbf{x}_1}}$ and $\mathbf{X}_2 \in \Re^{n_{\mathbf{x}_2}}$, i.e., $\mathbf{X} = \{\mathbf{X}_1, \mathbf{X}_2\} \in \Re^{n_\mathbf{x}} = \Re^{n_{\mathbf{x}_1}} \times \Re^{n_{\mathbf{x}_2}}$. For analyzing dynamic systems governed by second order stochastic differential equations, for instance, $\mathbf{X}_1$ contains pairs of displacement and its first derivative or corresponding velocity.

Denote the PDF of $\mathbf{X}_1$ as $p_1(\mathbf{x}_1,t)$. In order to obtain the $p_1(\mathbf{x}_1,t)$, integrating Eq. (5) over $\Re^{n_{\mathbf{x}_2}}$ gives

$$\begin{aligned}\int_{\Re^{n_{\mathbf{x}_2}}} \frac{\partial p(\mathbf{x},t)}{\partial t} d\mathbf{x}_2 &+ \int_{\Re^{n_{\mathbf{x}_2}}} \frac{\partial}{\partial x_j}\left[f_j(\mathbf{x})p(\mathbf{x},t)\right] d\mathbf{x}_2 \\ &- \frac{1}{2}\int_{\Re^{n_{\mathbf{x}_2}}} \frac{\partial^2}{\partial x_i \partial x_j}\left[G_{ij}(\mathbf{x})p(\mathbf{x},t)\right] d\mathbf{x}_2 = 0.\end{aligned} \tag{9}$$

Because of Eq. (8), we have

$$\int_{\Re^{n_{\mathbf{x}_2}}} \frac{\partial}{\partial x_j}\left[f_j(\mathbf{x})p(\mathbf{x},t)\right] d\mathbf{x}_2 = 0 \qquad x_j \in \Re^{n_{\mathbf{x}_2}} \tag{10}$$

and

$$\int_{\Re^{n_{\mathbf{x}_2}}} \frac{\partial^2}{\partial x_i \partial x_j}\left[G_{ij}(\mathbf{x})p(\mathbf{x},t)\right] d\mathbf{x}_2 = 0 \qquad x_i \text{ or } x_j \in \Re^{n_{\mathbf{x}_2}}. \tag{11}$$

Equation (9) can then be expressed as

$$\begin{aligned}\int_{\Re^{n_{\mathbf{x}_2}}} \frac{\partial p(\mathbf{x},t)}{\partial t} d\mathbf{x}_2 &+ \int_{\Re^{n_{\mathbf{x}_2}}} \frac{\partial}{\partial x_j}\left[f_j(\mathbf{x})p(\mathbf{x},t)\right] d\mathbf{x}_2 \\ &- \frac{1}{2}\int_{\Re^{n_{\mathbf{x}_2}}} \frac{\partial^2}{\partial x_i \partial x_j}\left[G_{ij}(\mathbf{x})p(\mathbf{x},t)\right] d\mathbf{x}_2 = 0 \qquad x_i, x_j \in \Re^{n_{\mathbf{x}_1}}\end{aligned} \tag{12}$$

which can be further expressed as

$$\begin{aligned}&\frac{\partial}{\partial t}\int_{\Re^{n_{\mathbf{x}_2}}} p(\mathbf{x},t)d\mathbf{x}_2 + \frac{\partial}{\partial x_j}\left[\int_{\Re^{n_{\mathbf{x}_2}}} f_j(\mathbf{x})p(\mathbf{x},t)d\mathbf{x}_2\right]\\ &\quad - \frac{1}{2}\frac{\partial^2}{\partial x_i \partial x_j}\left[\int_{\Re^{n_{\mathbf{x}_2}}} G_{ij}(\mathbf{x})p(\mathbf{x},t)d\mathbf{x}_2\right] = 0 \qquad x_i, x_j \in \Re^{n_{\mathbf{x}_1}}\end{aligned} \tag{13}$$

Separate $f_j(\mathbf{x})$ and $G_{ij}(\mathbf{x})$ into two parts, respectively, as

$$f_j(\mathbf{x}) = f_j^I(\mathbf{x}_1) + f_j^{II}(\mathbf{x}) \tag{14}$$

$$G_{ij}(\mathbf{x}) = G_{ij}^I(\mathbf{x}_1) + G_{ij}^{II}(\mathbf{x}) \tag{15}$$

Substituting Eqs. (14) and (15) into Eq. (13) gives

$$\begin{aligned}&\frac{\partial p_1(\mathbf{x}_1,t)}{\partial t} + \frac{\partial}{\partial x_j}\left[f_j^I(\mathbf{x}_1)p_1(\mathbf{x}_1,t) + \int_{\Re^{n_{\mathbf{x}_2}}} f_j^{II}(\mathbf{x})p(\mathbf{x},t)d\mathbf{x}_2\right]\\ &\quad - \frac{1}{2}\frac{\partial^2}{\partial x_i \partial x_j}\left[G_{ij}^I(\mathbf{x}_1)p_1(\mathbf{x}_1,t) + \int_{\Re^{n_{\mathbf{x}_2}}} G_{ij}^{II}(\mathbf{x})p(\mathbf{x},t)d\mathbf{x}_2\right] = 0\\ &\qquad (x_i, x_j \in \Re^{n_{\mathbf{x}_1}})\end{aligned} \tag{16}$$

For systems, normally $f_j^{II}(\mathbf{x})$ and $G_{ij}^{II}(\mathbf{x})$ are functions of only few state variables. Denote $f_j^{II}(\mathbf{x}) = f_j^{II}(\mathbf{x}_1, \mathbf{z}_k)$ in which $\mathbf{z}_k \in \Re^{n_{\mathbf{z}_k}} \subset \Re^{n_{\mathbf{x}_2}}$, and $G_{ij}^{II}(\mathbf{x}) = G_{ij}^{II}(\mathbf{x}_1, \mathbf{z}_r)$ in which $\mathbf{z}_r \in \Re^{n_{\mathbf{z}_r}} \subset \Re^{n_{\mathbf{x}_2}}$. $n_{\mathbf{z}_k}$ is the number of the state variables in $\mathbf{z}_k$ and $n_{\mathbf{z}_r}$ is the number of the state variables in $\mathbf{z}_r$. Therefore, Eq. (16) can be expressed as

$$\begin{aligned}&\frac{\partial p_1(\mathbf{x}_1,t)}{\partial t} + \frac{\partial}{\partial x_j}\left[f_j^I(\mathbf{x}_1)p_1(\mathbf{x}_1,t) + \int_{\Re^{n_{\mathbf{z}_k}}} f_j^{II}(\mathbf{x}_1,\mathbf{z}_k)p_k(\mathbf{x}_1,\mathbf{z}_k,t)d\mathbf{z}_k\right]\\ &\quad - \frac{1}{2}\frac{\partial^2}{\partial x_i \partial x_j}\left[G_{ij}^I(\mathbf{x}_1)p_1(\mathbf{x}_1,t) + \int_{\Re^{n_{\mathbf{z}_r}}} G_{ij}^{II}(\mathbf{x}_1,\mathbf{z}_r)p_r(\mathbf{x}_1,\mathbf{z}_r,t)d\mathbf{z}_r\right] = 0\\ &\qquad (x_i, x_j \in \Re^{n_{\mathbf{x}_1}})\end{aligned} \tag{17}$$

in which $p_k(\mathbf{x}_1, \mathbf{z}_k, t)$ denotes the joint PDF of $\{\mathbf{X}_1, \mathbf{Z}_k\}$ and $p_r(\mathbf{x}_1, \mathbf{z}_r, t)$ denotes the joint PDF of $\{\mathbf{X}_1, \mathbf{Z}_r\}$. The summation convention not applies on the indexes k and r in Eq. (17) and in the following discussions.

From Eq. (17), it is seen that the coupling of $\mathbf{X}_1$ and $\mathbf{X}_2$ comes from $f_j^{II}(\mathbf{x}_1, \mathbf{z}_k)p_k(\mathbf{x}_1, \mathbf{z}_k, t)$ and $G_{ij}^{II}(\mathbf{x}_1, \mathbf{z}_r)p_r(\mathbf{x}_1, \mathbf{z}_r, t)$. Express $p_k(\mathbf{x}_1, \mathbf{z}_k, t)$ as

$$p_k(\mathbf{x}_1, \mathbf{z}_k, t) = p_1(\mathbf{x}_1, t)q_k(\mathbf{z}_k, t; \mathbf{x}_1) \tag{18}$$

where $q_k(\mathbf{z}_k, t; \mathbf{x}_1)$ is the conditional PDF of $\mathbf{Z}_k$ for given $\mathbf{X}_1 = \mathbf{x}_1$, and express $p_k(\mathbf{x}_1, \mathbf{z}_r, t)$ as

$$p_r(\mathbf{x}_1, \mathbf{z}_r, t) = p_1(\mathbf{x}_1, t)q_r(\mathbf{z}_r, t; \mathbf{x}_1) \tag{19}$$

where $q_r(\mathbf{z}_r, t; \mathbf{x}_1)$ is the conditional PDF of $\mathbf{Z}_r$ for given $\mathbf{X}_1 = \mathbf{x}_1$.

Substituting Eqs. (18) and (19) into Eq. (17) gives

$$\frac{\partial p_1(\mathbf{x}_1,t)}{\partial t}+\frac{\partial}{\partial x_j}\left\{\left[f_j^I(\mathbf{x}_1)+\int_{\Re^{n_{\mathbf{z}_k}}} f_j^{II}(\mathbf{x}_1,\mathbf{z}_k)q_k(\mathbf{z}_k,t;\mathbf{x}_1)d\mathbf{z}_k\right]p_1(\mathbf{x}_1,t)\right\}$$
$$-\frac{1}{2}\frac{\partial^2}{\partial x_i\partial x_j}\left\{\left[G_{ij}^I(\mathbf{x}_1)+\int_{\Re^{n_{\mathbf{z}_r}}} G_{ij}^{II}(\mathbf{x}_1,\mathbf{z}_r)q_r(\mathbf{z}_r,t;\mathbf{x}_1)d\mathbf{z}_r\right]p_1(\mathbf{x}_1,t)\right\}=0$$
$$(x_i,x_j\in\Re^{n_{\mathbf{x}_1}}) \tag{20}$$

Assume that the approximate probabilistic solutions of the NSD systems are obtainable with the EQL method. Then approximately replacing the conditional PDFs $q_k(\mathbf{z}_k,t;\mathbf{x}_1)$ and $q_r(\mathbf{z}_r,t;\mathbf{x}_1)$ by those from EQL, Eq (20) can be expressed as

$$\frac{\partial \tilde{p}_1(\mathbf{x}_1,t)}{\partial t}+\frac{\partial}{\partial x_j}\left\{\left[f_j^I(\mathbf{x}_1)+\int_{\Re^{n_{\mathbf{z}_k}}} f_j^{II}(\mathbf{x}_1,\mathbf{z}_k)\overline{q}_k(\mathbf{z}_k,t;\mathbf{x}_1)d\mathbf{z}_k\right]\tilde{p}_1(\mathbf{x}_1,t)\right\}$$
$$-\frac{1}{2}\frac{\partial^2}{\partial x_i\partial x_j}\left\{\left[G_{ij}^I(\mathbf{x}_1)+\int_{\Re^{n_{\mathbf{z}_r}}} G_{ij}^{II}(\mathbf{x}_1,\mathbf{z}_r)\overline{q}_r(\mathbf{z}_r,t;\mathbf{x}_1)d\mathbf{z}_r\right]\tilde{p}_1(\mathbf{x}_1,t)\right\}=0$$
$$(x_i,x_j\in\Re^{n_{\mathbf{x}_1}}) \tag{21}$$

where $\overline{q}_k(\mathbf{z}_k,t;\mathbf{x}_1)$ is the conditional PDF of $\mathbf{Z}_k$ from EQL for given $\mathbf{X}_1=\mathbf{x}_1$, $\overline{q}_r(\mathbf{z}_r,t;\mathbf{x}_1)$ is the conditional PDF of $\mathbf{Z}_r$ from EQL for given $\mathbf{X}_1=\mathbf{x}_1$, and $\tilde{p}_1(\mathbf{x}_1,t)$ is the approximate PDF of $\mathbf{X}_1$. Denote

$$\tilde{f}_j(\mathbf{x}_1,t)=f_j^I(\mathbf{x}_1)+\int_{\Re^{n_{\mathbf{z}_k}}} f_j^{II}(\mathbf{x}_1,\mathbf{z}_k)\overline{q}_k(\mathbf{z}_k,t;\mathbf{x}_1)d\mathbf{z}_k \tag{22}$$

$$\tilde{G}_{ij}(\mathbf{x}_1,t)=G_{ij}^I(\mathbf{x}_1)+\int_{\Re^{n_{\mathbf{z}_r}}} G_{ij}^{II}(\mathbf{x}_1,\mathbf{z}_r)\overline{q}_r(\mathbf{z}_r,t;\mathbf{x}_1)d\mathbf{z}_r \tag{23}$$

Then Eq. (21) can be expressed as

$$\frac{\partial \tilde{p}_1(\mathbf{x}_1,t)}{\partial t}+\frac{\partial}{\partial x_j}\left[\tilde{f}_j(\mathbf{x}_1,t)\tilde{p}_1(\mathbf{x}_1,t)\right]-\frac{1}{2}\frac{\partial^2}{\partial x_i\partial x_j}\left[\tilde{G}_{ij}(\mathbf{x}_1,t)\tilde{p}_1(\mathbf{x}_1,t)\right]=0\quad(x_i,x_j\in\Re^{n_{\mathbf{x}_1}}) \tag{24}$$

which is the approximate FP equation governing the approximate joint PDF of the state variables in the sub state space $\Re^{n_{\mathbf{x}_1}}$. In the stationary state, Eq. (24) is reduced to

$$\frac{\partial}{\partial x_j}\left[\tilde{f}_j(\mathbf{x}_1)\tilde{p}_1(\mathbf{x}_1)\right]-\frac{1}{2}\frac{\partial^2}{\partial x_i\partial x_j}\left[\tilde{G}_{ij}(\mathbf{x}_1)\tilde{p}_1(\mathbf{x}_1)\right]=0\quad(x_i,x_j\in\Re^{n_{\mathbf{x}_1}}) \tag{25}$$

which is the approximate reduced FP equation governing the approximate joint PDF $\tilde{p}_1(\mathbf{x}_1)$ of the stationary state variables in the sub state space $\Re^{n_{\mathbf{x}_1}}$.

If $\mathbf{X}_1$ only contains few state variables, the EPC method can be employed to solve Eq. (24) or (25) [29,30]. Therefore, the whole solution procedure may be named SSS-EPC method for short in the following discussions.

4 Examples

From the above discussion, it is seen that the SSS procedure is not limited by the number of state variables in the NSD systems. Four systems are analyzed with the above solution procedure in the following discussions. The first example is about a 10-DOF NSD system with additive excitations. There are 20 state variables in this system. The second example is same as the first one except that multiplicative excitations on displacements are added to the system. The third example is about a 8-DOF NSD system with both additive and multiplicative excitations on velocities. The fourth example is about the nonlinear random vibration of a flexural beam supported by nonlinear springs and excited by white noise. Only part of the PDFs and logarithmic PDFs of the state variables are presented because the limited paper size. The results obtained with the SSS-EPC method are compared with those from MCS and EQL to verify the effectiveness of the presented SSS procedure. The sample size is 10^7 in MCS. In the presented figures, the results corresponding to $n = 4$ are the results obtained from SSS-EPC when the polynomial degree equals 4 in the EPC solution procedure, σ_{y_i} denotes the standard deviation of Y_i from EQL, and $\sigma_{\dot{y}_i}$ denotes the standard deviation of $\dot{Y}_i$ from EQL.

4.1 10-degree-of-freedom system with additive excitations

Consider the following 10-degree-of-freedom system with high nonlinearity:

$$\ddot{\mathbf{Y}}(t) + \mathbf{C}\dot{\mathbf{Y}}(t) + \mathbf{KY} + \mathbf{H}(\mathbf{Y}, \dot{\mathbf{Y}}) = \mathbf{F}(t) \tag{26}$$

in which $\mathbf{Y}(t) = \{Y_1, Y_2, \cdots, Y_{10}\}^t$; $\mathbf{H}(\mathbf{Y}, \dot{\mathbf{Y}}) = \{Y_1^3 + 0.3\dot{Y}_1^3, Y_2^3 + 0.3\dot{Y}_2^3, \cdots, Y_{10}^3 + 0.3\dot{Y}_{10}^3\}^t$; $\mathbf{F}(t) = \{1, 1, \cdots, 1\}^t W(t)$; $W(t)$ is a white noise with unit power spectral density; and

$$\mathbf{K} = \begin{bmatrix} 1 & 0.2 & -0.1 & 0.3 & 0.2 & -0.2 & 0.1 & 0 & 0 & 0 \\ 0.2 & 1 & 0.2 & 0.3 & 0.2 & 0.1 & 0.3 & 0 & 0 & 0 \\ -0.1 & 0.2 & 1 & 0.2 & 0.1 & 0.2 & -0.2 & 0 & 0 & 0 \\ 0.3 & 0.3 & 0.2 & 1 & 0.2 & 0.2 & 0.1 & 0.3 & 0.2 & 0.1 \\ 0.2 & 0.2 & 0.1 & 0.2 & 1 & 0.1 & 0.3 & 0.2 & 0.2 & 0.2 \\ -0.2 & 0.1 & 0.2 & 0.2 & 0.1 & 1 & 0.5 & 0.3 & 0.3 & 0.3 \\ 0.1 & 0.3 & -0.2 & 0.1 & 0.3 & 0.5 & 1 & 0.2 & -0.3 & 0.1 \\ 0 & 0 & 0 & 0.3 & 0.2 & 0.3 & 0.2 & 1 & 0.2 & 0.1 \\ 0 & 0 & 0 & 0.2 & 0.2 & 0.3 & -0.3 & 0.2 & 1 & 0.3 \\ 0 & 0 & 0 & 0.1 & 0.2 & 0.3 & 0.1 & 0.1 & 0.3 & 1 \end{bmatrix} \tag{27}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0.2 & 0.1 & -0.1 & 0.3 & 0.2 & 0.1 & 0 & 0 & 0 \\ 0.2 & 1 & 0.2 & 0.2 & 0.1 & 0.2 & 0.1 & 0 & 0 & 0 \\ 0.1 & 0.2 & 1 & 0.1 & 0.1 & 0.2 & 0.2 & 0 & 0 & 0 \\ -0.1 & 0.2 & 0.1 & 1 & 0.1 & 0.3 & -0.2 & 0.4 & 0.2 & 0.2 \\ 0.3 & 0.1 & 0.1 & 0.1 & 1 & 0.1 & -0.2 & 0.3 & -0.4 & 0.1 \\ 0.2 & 0.2 & 0.2 & 0.3 & 0.1 & 1 & 0.3 & 0.2 & 0.1 & 0.2 \\ 0.1 & 0.1 & 0.2 & -0.2 & -0.2 & 0.3 & 1 & 0.1 & 0.2 & 0.3 \\ 0 & 0 & 0 & 0.4 & 0.3 & 0.2 & 0.1 & 1 & 0.3 & 0.1 \\ 0 & 0 & 0 & 0.2 & -0.4 & 0.1 & 0.2 & 0.3 & 1 & 0.2 \\ 0 & 0 & 0 & 0.2 & 0.1 & 0.2 & 0.3 & 0.1 & 0.2 & 1 \end{bmatrix} \tag{28}$$

The stationary PDFs obtained with the SSS-EPC method, MCS, and EQL methods are compared in order to show the effectiveness of the SSS-EPC method in analyzing the large-scale highly nonlinear stochastic dynamic systems with additive excitations. With the SSS-EPC method, the stationary PDFs $p_1(\mathbf{X}_{1i})$ are obtained by taking $\mathbf{X}_{1i} = \{Y_i, \dot{Y}_i\}$. Only the PDFs and logarithmic PDFs of Y_5 and $\dot{Y}_5$ are shown and compared in Figs. 1(a–d). It is seen in the figures that the PDFs and the tails of the PDFs of Y_5 and $\dot{Y}_5$ obtained with SSS-EPC are close to MCS while the PDFs from EQL method deviate much

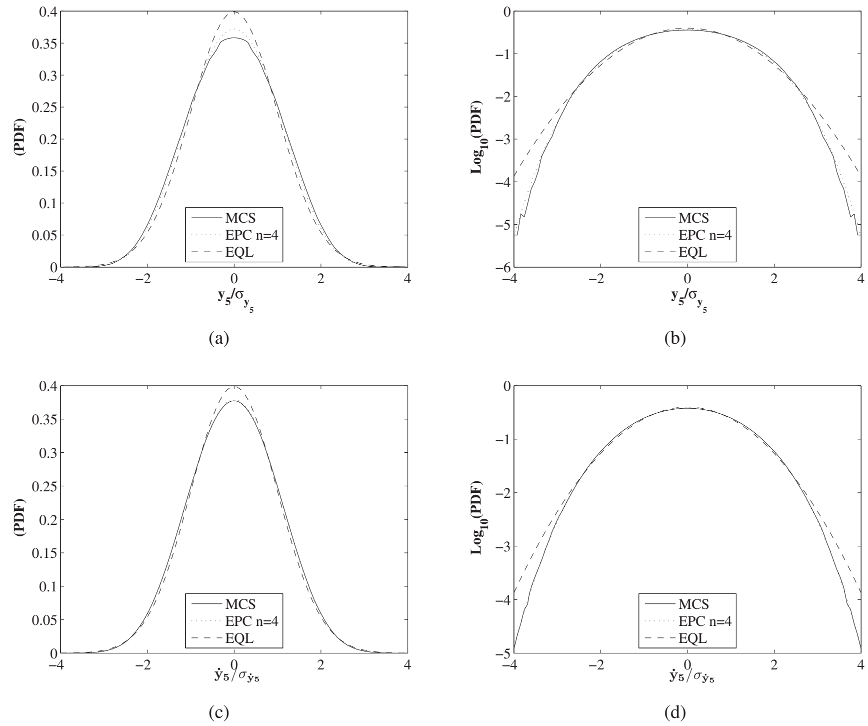

Fig. 1 Comparison of PDFs and logarithmic PDFs in example 1: (a) PDFs of displacement Y_5; (b) logarithmic PDFs of displacement Y_5; (c) PDFs of velocity $\dot{Y}_5$; (d) Logarithmic PDFs of velocity $\dot{Y}_5$.

from simulation, especially in the tail regions which play an important role in dynamic system reliability analysis. Similar behavior of the PDFs and the tails of the PDFs of other state variables can also be observed without being presented here.

4.2 10-degree-of-freedom system with multiplicative excitations on displacements

Consider the following 10-degree-of-freedom system with high nonlinearity and both additive and multiplicative excitations on displacements:

$$\ddot{\mathbf{Y}}(t) + \mathbf{C}\dot{\mathbf{Y}}(t) + \mathbf{K}\mathbf{Y} + \mathbf{H}(\mathbf{Y}, \dot{\mathbf{Y}}) = \mathbf{F}(\mathbf{Y}, t) \tag{29}$$

This system is same as that in the above example except $\mathbf{F}(\mathbf{Y}, t) = \{W_1(t) - 0.3Y_1W_2(t), W_1(t) - 0.3Y_2W_2(t), \cdots, W_1(t) - 0.3Y_{10}W_2(t)\}^t$ in which $W_1(t)$ and $W_2(t)$ are independent Gaussian white noises with unit power spectral density.

The stationary PDFs obtained with the SSS-EPC, MCS, and EQL are compared in order to further show the effectiveness of the SSS-EPC method in analyzing the large-scale NSD systems with both additive and multiplicative excitations on displacements. Similar to the above example, the PDF solutions are also obtained by taking $\mathbf{X}_{1i} = \{Y_i, \dot{Y}_i\}$ in obtaining the PDFs $p_1(\mathbf{x}_{1i})$ with the SSS-EPC method. Only the PDFs and logarithmic PDFs of Y_5 and $\dot{Y}_5$ are shown and compared in Figs. 2(a–d). It is seen from Figs. 2(a–d) that the PDFs and the tails of the PDFs of Y_5 and $\dot{Y}_5$ obtained from SSS-EPC when the polynomial order equals 4 in the EPC procedure are still close to MCS. On the other hand, the PDFs from EQL method deviate much from simulation. Similar behavior of the PDFs and the tails of the PDFs of other state variables can also be observed without being presented here.

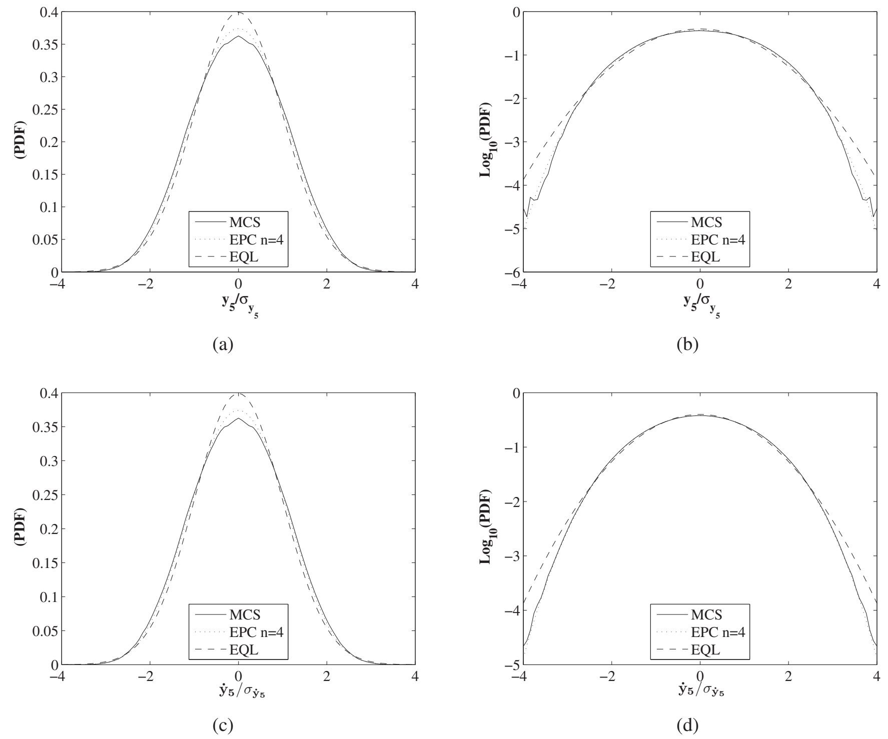

Fig. 2 Comparison of PDFs and logarithmic PDFs in example 2: (a) PDFs of displacement Y_5; (b) logarithmic PDFs of displacement Y_5; (c) PDFs of velocity $\dot{Y}_5$; (d) logarithmic PDFs of velocity $\dot{Y}_5$.

4.3 8-degree-of-freedom system with multiplicative excitations on velocities

Consider the following 8-degree-of-freedom nonlinear system with additive excitations and multiplicative excitations on velocities:

$$\ddot{\mathbf{Y}}(t) + \mathbf{C}\dot{\mathbf{Y}}(t) + \mathbf{K}\mathbf{Y} + \mathbf{H}(\mathbf{Y}, \dot{\mathbf{Y}}) = \mathbf{F}(\dot{\mathbf{Y}}, t) \tag{30}$$

in which $\mathbf{Y}(t) = \{Y_1, Y_2, \cdots, Y_8\}^t$; $\mathbf{H}(\mathbf{Y}, \dot{\mathbf{Y}}) = \{Y_1^3 + 0.5\dot{Y}_1^3, Y_2^3 + 0.5\dot{Y}_2^3, \cdots, Y_8^3 + 0.5\dot{Y}_8^3\}^t$; $\mathbf{F}(\dot{\mathbf{Y}}, t) = \{W_1(t) - 0.3\dot{Y}_1 W_2(t), W_1(t) - 0.3\dot{Y}_2 W_2(t), \cdots, W_1(t) - 0.3\dot{Y}_8 W_2(t)\}^t$; $W_1(t)$ and $W_2(t)$ are the white noises with unit power spectral density; and

$$\mathbf{K} = \begin{bmatrix} 1 & 0.2 & -0.1 & 0.3 & 0.5 & -0.2 & 0.1 & 0 \\ 0.2 & 1 & 0.2 & 0.3 & -0.2 & 0.1 & 0.3 & 0 \\ -0.1 & 0.2 & 1 & 0.2 & 0.1 & 0.2 & -0.2 & 0 \\ 0.3 & 0.3 & 0.2 & 1 & 0.2 & 0.2 & 0.1 & 0.3 \\ 0.5 & -0.2 & 0.1 & 0.2 & 1 & 0.1 & 0.3 & 0.2 \\ -0.2 & 0.1 & 0.2 & 0.2 & 0.1 & 1 & 0.5 & -0.3 \\ 0.1 & 0.3 & -0.2 & 0.1 & 0.3 & 0.5 & 1 & 0.2 \\ 0 & 0 & 0 & 0.3 & 0.2 & -0.3 & 0.2 & 1 \end{bmatrix} \tag{31}$$

$$
\mathbf{C} = \begin{bmatrix}
1 & 0.2 & 0.4 & -0.1 & 0.3 & 0.2 & 0 & 0 \\
0.2 & 1 & 0.2 & 0.2 & 0.1 & 0.2 & 0 & 0 \\
0.4 & 0.2 & 1 & 0.1 & 0.1 & 0.2 & 0.2 & 0 \\
-0.1 & 0.2 & 0.1 & 1 & 0.1 & 0.3 & -0.2 & 0.4 \\
0.3 & 0.1 & 0.1 & 0.1 & 1 & 0.1 & -0.2 & 0.3 \\
0.2 & 0.2 & 0.2 & 0.3 & 0.1 & 1 & 0.3 & 0.2 \\
0 & 0 & 0.2 & -0.2 & -0.2 & 0.3 & 1 & 0.1 \\
0 & 0 & 0 & 0.4 & 0.3 & 0.2 & 0.1 & 1
\end{bmatrix} \tag{32}
$$

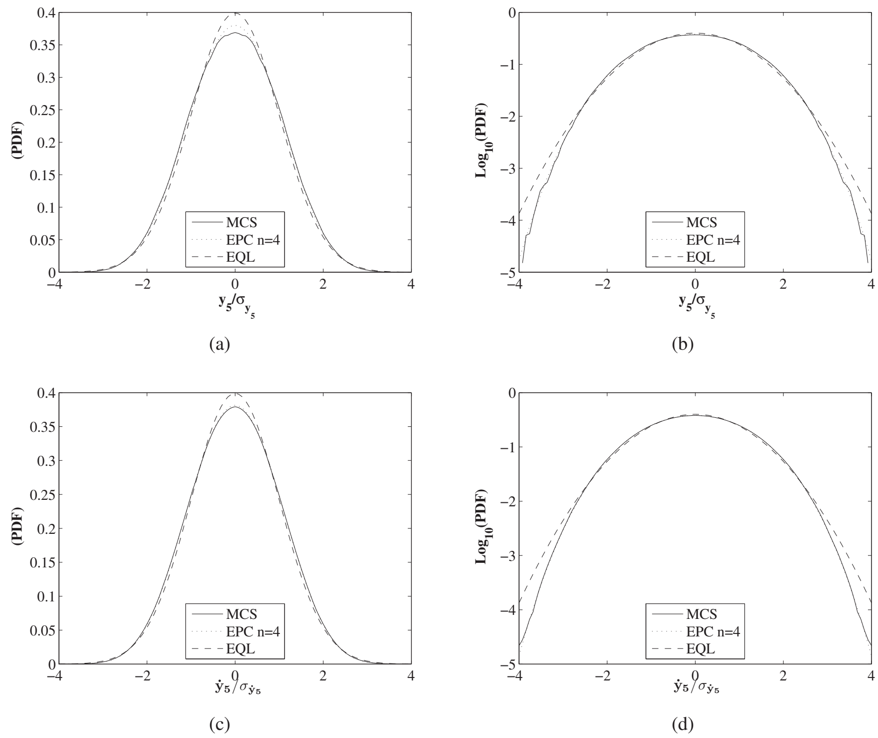

Fig. 3 Comparison of PDFs and logarithmic PDFs in example 3: (a) PDFs of displacement Y_5; (b) Logarithmic PDFs of displacement Y_5; (c) PDFs of velocity $\dot{Y}_5$; (d) logarithmic PDFs of velocity $\dot{Y}_5$.

The stationary PDFs obtained with the SSS-EPC, MCS, and EQL are compared in order to further show the effectiveness of the SSS-EPC method in analyzing the large-scale NSD systems with both additive and multiplicative excitations on velocities. With the SSS-EPC method, the PDF solutions are also obtained by taking $\mathbf{X}_{1i} = \{Y_i, \dot{Y}_i\}$ in obtaining the PDFs $p_1(\mathbf{x}_{1i})$. It is seen from Figs. 3(a–d) that the PDFs and the tails of the PDFs of Y_5 and $\dot{Y}_5$ obtained from SSS-EPC when the polynomial order equals 4 in the EPC procedure are also close to MCS in this case, specially in the tails of the PDFs as shown by the logarithmic PDFs. On the other hand, the PDFs and the tails of the PDFs obtained from EQL deviate much from simulation. Similar behavior of the PDFs and the tails of the PDFs of other state variables can also be observed without being presented here.

4.4 Random vibration of the flexural beam supported by nonlinear springs

Consider a flexural steel beam with pin support at one end and roller support at another, supported by nonlinear springs, and excited by point loads being white noise. The mechanical model for vibrational analysis is shown in Fig. 4.

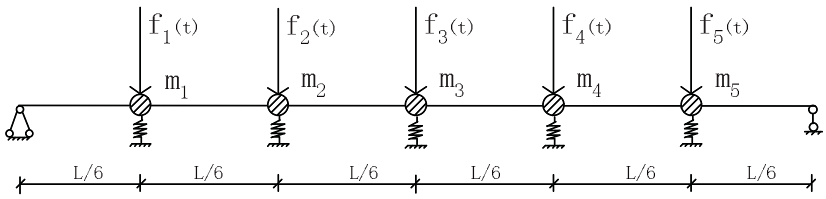

Fig. 4 Dynamic model for the flexural beam supported by nonlinear springs.

The vertical displacement of mass m_i is denoted as Y_i. The beam length is 5 m, The young's modulus is 2.1×10^{11} Pa, The area of the cross section of the beam is 8.61×10^{-3} m^2, and the moment inertia of the cross section is 2.17×10^{-4} m^4. The mass density of the beam material is 7850 kg/m^2. Then the governing equations for the vertical displacements can be obtained with flexibility method as follows.

$$\ddot{\mathbf{Y}}(t) + \mathbf{C}\dot{\mathbf{Y}}(t) + \mathbf{K}\mathbf{Y} + \mathbf{H}(\mathbf{Y}) = \mathbf{F}(t) \tag{33}$$

in which $\mathbf{Y}(t)$ =$\{Y_1, Y_2, \cdots, Y_5\}^t$; $\mathbf{H}(\mathbf{Y})$=$3 \times 10^6\{Y_1^3, Y_2^3, \cdots, Y_5^3\}^t$; $\mathbf{F}(t)$=$1000\{1, 1, \cdots, 1\}^t W(t)$; $W(t)$ is a white noise with unit power spectral density; and

$$\mathbf{K} = \begin{bmatrix} 138.07 & -132.91 & 58.066 & -15.484 & 3.8711 \\ -132.91 & 196.13 & -148.39 & 61.937 & -15.484 \\ 58.066 & -148.39 & 200.00 & -148.39 & 58.066 \\ -15.484 & 61.937 & -148.39 & 196.13 & -132.91 \\ 3.8711 & -15.484 & 58.066 & -132.91 & 138.07 \end{bmatrix} \times 10^5 \tag{34}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \times 10^3 \tag{35}$$

This system is analyzed with the above SSS-EPC procedure and Monte Carlo simulation for comparison. Under the action of the excitations, the maximum vertical displacement of the beam is Y_3 which is concerned in system design. The PDF and the logarithmic PDFs of Y_3 are shown in Figs. 5(a) and (b), respectively. It is observed that the PDFs and the tails of the PDFs of Y_3 obtained from SSS-EPC when the polynomial order equals 4 in the EPC procedure are also close to MCS for this system, specially in the tails of the PDFs as shown by the logarithmic PDFs. On the other hand, the PDFs and the tails of the PDFs obtained from EQL deviate much from simulation. In practice, the mean up-crossing rate (MCR) of the system response is frequently used for system reliability analysis. The MCR at $Y_i = y_i$ is defined as

$$\nu^+(y_i) = \int_0^{+\infty} \dot{y}_i p(y_i, \dot{y}_i) d\dot{y}_i \tag{36}$$

where $p(y_i, \dot{y}_i)$ denotes the joint PDF of Y_i and $\dot{Y}_i$, and $\nu^+(y_i)$ denotes the MCR at $Y_i = y_i$.

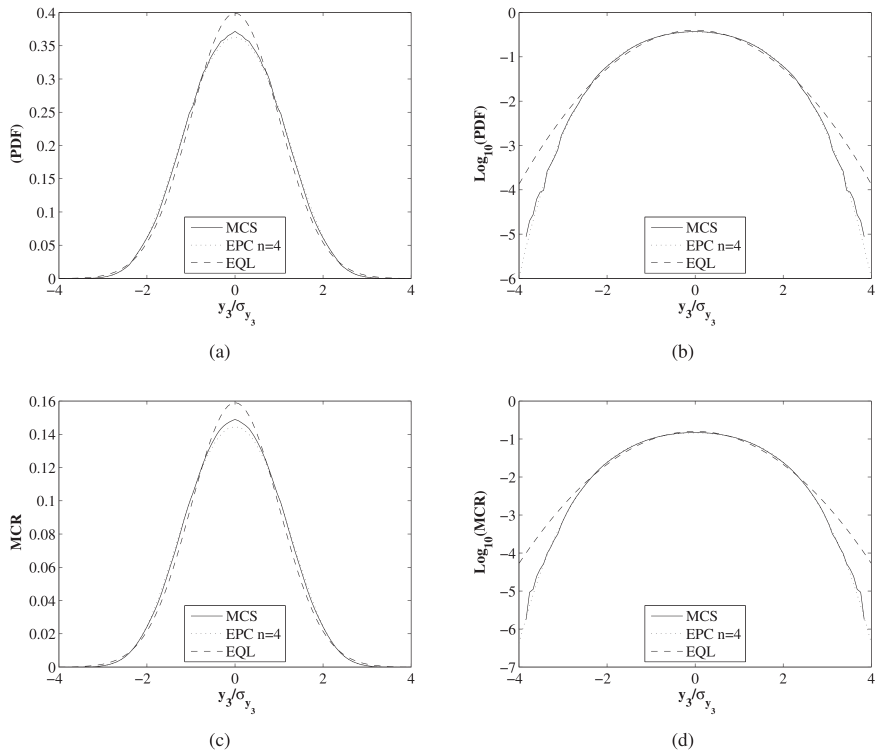

Fig. 5 Comparison of PDFs, logarithmic PDFs, MCRs, and logarithmic MCRs in example 4: (a) PDFs of displacement Y_3; (b) logarithmic PDFs of displacement Y_3; (c) MCRs at $Y_3 = y_3$; (c) logarithmic MCRs at $Y_3 = y_3$.

The MCRs and the logarithmic MCRs of Y_3 are shown and compared in Figs. 5(c) and (d), respectively. It is still observed that the MCRs and the tails of the MCRs of Y_3 obtained from SSS-EPC when the polynomial order equals 4 in the EPC procedure are close to MCS for this system, specially in the tails of the MCRs as shown by the logarithmic MCRs, which is important for system reliability analysis. On the other hand, the MCRs and the tails of the MCRs obtained from EQL deviate much from simulation.

The presented solution procedure and the numerical results presented in the above four examples have demonstrated that the SSS-EPC method is not limited by the number of the state variables in the systems, the level of system nonlinearity, and the existence of multiplicative excitations on displacement or velocities. Because the problem of solving the FP equation in high-dimensional state space becomes the problem of solving some FP equations in low-dimensional state spaces or even two-dimensional state space, the required computational effort is very small.

5 Conclusions

A new methodology named state space split is presented in this paper to solve the reduced FP equation in high-dimensional state space. With this method, the state space of NSD system is split into two subspaces. The reduced FP equation is integrated in one of the subspaces and the reduced FP equation in another subspace is derived which governs the approximate joint PDF of the state variables in the subspace. Therefore, the problem of solving the reduced FP equation in high-dimensional state space becomes the problem of solving some reduced FP equations in low-dimension state spaces. The EPC method can then be employed to solve the reduced FP equation in the low-dimension state spaces. The whole solution procedure is named SSS-EPC method. This method is not limited by the number of state variables of the systems. Therefore it can be employed for analyzing the probabilistic solutions of large-scale NSD systems. Numerical results have shown that the PDFs and logarithmic PDFs of the systems responses obtained with the SSS-EPC method are close to Monte Carlo simulation. The tails of the PDFs obtained from the SSS-EPC method also behave well. This new method is not only suitable for the large-scale NSD systems with slight nonlinearity, but also suitable for the systems with high nonlinearity. The PDF solution of large-scale NSD system with both additive and multiplicative excitations on either displacements or velocities can also be analyzed accurately with the SSS-EPC method. It attempts to provide an analytical tool for the accurate probabilistic solutions of many NSD systems in statistical physics and other areas of science and engineering. It is also seen that this method only works for the systems which can be analyzed with EQL method because the result obtained from EQL is needed in the SSS-EPC solution procedure.

Acknowledgements The results presented in this paper are obtained with the support from the Research and Development Administration Office of the University of Macau.

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