SSS-EPC Method for the Transient Probabilistic Solutions of Multi-Degree-of-Freedom Nonlinear Stochastic Dynamical Systems

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Summary. The state-space-split (SSS) method and exponential-polynomial-closure (EPC) method have been jointly applied to analyze the stationary probabilistic solutions of various multi-degree-of-freedom (MDOF) nonlinear stochastic dynamical (NSD) systems in the last over ten years [1-5]. In this paper, the SSS-EPC method is extended to analyze the transient probabilistic solutions of MDOF-NSD systems under the excitation of Gaussian white noise.

The MDOF-NSD system and FPK equation

The random vibration of real-world problems is usually described by the following MDOF-NSD system.

$$\frac{d}{dt}X_i = f_i(\mathbf{X},t) + g_{ij}(\mathbf{X},t)W_j(t) \qquad (i = 1,2,\ldots,n_x; j = 1,2,...,n_w) \tag{1}$$

where $\mathbf{X} \in \mathbb{R}^{n_x}$ is the vector of state variables and n_x is the number of state variables, $W_j(t)$ is the jth Gaussian white noise and n_w is the number of white noises. The summation convention applies in the Eqs. (1) and (2). The transient probability density function (PDF) $p(\mathbf{x},t)$ of the state vector $\mathbf{X}$ is governed by the following FPK equation.

$$\frac{\partial p(\mathbf{x},t)}{\partial t} + \frac{\partial}{\partial x_j}\left[f_j(\mathbf{x},t)p(\mathbf{x},t)\right] - \frac{1}{2}\frac{\partial^2}{\partial x_i \partial x_j}\left[G_{ij}(\mathbf{x},t)p(\mathbf{x},t)\right] = 0 \tag{2}$$

where $G_{ij}(\mathbf{x},t) = S_{ls}g_{il}(\mathbf{x},t)g_{js}(\mathbf{x},t)$ with $i,j = 1,2,...,n_x$ and $l,s = 1,2,...,n_w$.
The SSS method can be adopted to make the dimension of the above FPK equation reduced [1]. After that, the dimension-reduced FPK equation can be solved by the EPC method for stationary [6] or transient [7] solutions.

Random vibration of von Karman plate

Without considering the static force due to gravitation, the governing equations of the random vibration of the von Karman plate are given as follows.

$$\ddot{w} + f(\dot{w}) + \frac{D}{r}\nabla^4 w - \frac{h}{r}(w_{,xx}F_{,yy} + w_{,yy}F_{,xx} - 2w_{,xy}F_{,xy}) = \frac{1}{h}W(t) \tag{3}$$

$$\nabla^4 F = h^2 E(w_{xy}^2 - w_{,xx}w_{,yy}) \tag{4}$$

where $w = w(x,y,t)$ is the ratio of the deflection and the thickness of the plate at point (x,y) and time t; $F = F(x,y,t)$ is the Airy stress function at point (x,y) and time t; $f(\dot{w})$ denotes the damping force which is formulated under the condition that the damping ratio for each mode is the same in the numerical analysis; $\dot{()} = \partial()/\partial t$; $\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$; $()_{,xx} = \partial^2()/\partial x^2$; $()_{,yy} = \partial^2()/\partial y^2$;$()_{,xy} = \partial^2()/\partial x\partial y$; $D = \frac{Eh^3}{12(1-\nu^2)}$; h denotes the thickness of plate; E denotes the Young's modulus; ν denotes the Poisson's ratio; $r = \rho h + q_w$ in which ρ denotes the mass density in kg/m^3 of the plate; q_w denotes the uniformly distributed mass in kg/m^2 applied on the plate; $W(t)$ denotes the vertical ground acceleration being Gaussian white noise with power spectral density S.

Numerical results

Consider the rectangular plate with length $a = 7m$ and width $b = 5m$, $h = 0.1m$, $E = 2.55 \times 10^{10} Pa$, $\nu = 0.316$, $\rho = 2,300 kg/m^3$, $q_w = 500 kg/m^2$ and $S = 1$. The damping ratio is 0.02 for each mode. The four edges of the plate are all simply supported and free-stressed in plane. With the finite difference grid as shown in Fig. 1, a $n_x \times n_y$-DOF NSD system can be formulated. With $n_x \times n_y = 3 \times 3$, a 9-DOF NSD system is formulated. In order to reduce the computational time needed by Monte Carlo simulation (MCS), only the system about 1/4 of the plate is formulated and analyzed according to the symmetry of the structure under the given conditions, which leads to a 4-DOF system. The results obtained by the SSS-EPC method, the equivalent linearization (EQL) method and the MCS method are compared. The probability density functions (PDFs) of the deflection at the center of the plate at time instant $t = 0.5s$ and $t = 1s$ are shown and compared in Fig. 2. The numerical analysis by all the three methods are on this 4-DOF system which corresponds to a 8-dimensional FPK equation. Even so, the computational time needed by the MCS with sample size being 2×10^6 is about 27,200 seconds and that needed by the SSS-EPC for each degree of freedom is about 20 seconds when the structure is analyzed from $0s$ to $1s$. It is noted that the basic period of the plate is $0.195s$. In order to show the tail behavior of the PDF solutions, the logarithmic values of the PDFs are also shown and compared in Fig. 2.

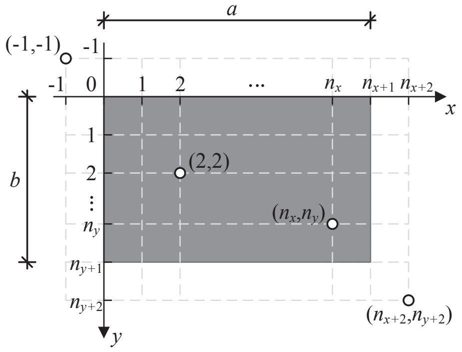
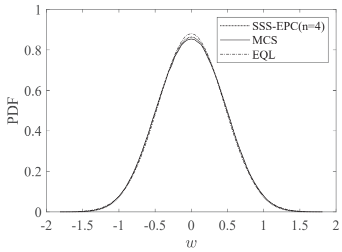
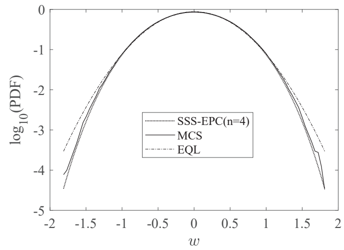


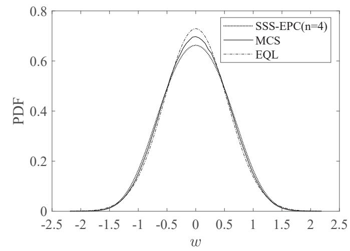
Figure 1: Finite difference grid of the rectangular plate



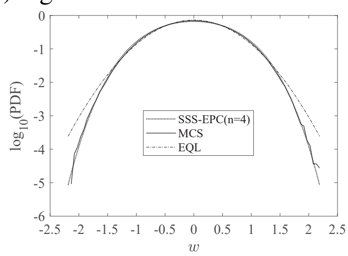
(a) PDF of w at $t = 0.5s$



(b) logarithmic PDF of w at $t = 0.5s$



(c) PDF of w at $t = 1s$



(d) logarithmic PDF of w at $t = 1s$

Figure 2: The probabilistic solutions of the deflection w in the middle of plate

Conclusions

Though the SSS procedure for the dimension reduction and transient solution of FPK equation was presented in [1], but it was only applied to analyze the stationary probabilistic solutions of some MDOF-NSD systems in the past. In this paper, the SSS method has been applied to analyze the transient probabilistic solution of MDOF-NSD system. As a detailed application, the MDOF-NSD system about the random vibration of von Karman plate is analyzed by the three available methods for analyzing the probabilistic solutions of large NSD systems, i.e., the Monte MCS method, EQL method and SSS-EPC method. The solution accuracy and computational efficiency of the methods are compared. Numerical analysis shows that the SSS-EPC method can give much more accurate solution than the EQL method and is much more efficient than MCS.

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