Phase transition in the EM scheme of an SDE driven by α-stable noises with $\alpha \in (0, 2]$

Lihu Xu

(Univeristy of Macau)

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Outline

This talk is based on the joint work with Yu Wang (PhD student at UM) and Yimin Xiao:

- Background
- Drift with polynomial growth+α-stable noise $(0 < \alpha < 2)$
- Critical drift $-x \log(1 + |x|)$: phase transition as $\alpha \uparrow 2$
- Simulation

SDE

The SDE on $\mathbb{R}^d$:

$$\mathrm{d}X_t = f(X_t)\,\mathrm{d}t + g(X_t)\,\mathrm{d}L_t, \quad X_0 = x_0 \in \mathbb{R}^d, \qquad (0.1)$$

where

- $f : \mathbb{R}^d \to \mathbb{R}^d$, $g : \mathbb{R}^d \to \mathbb{R}^{d \times d}$ satisfy certain regularity conditions,
- $(L_t, t \geqslant 0)$ is a d-dimensional, rotationally invariant α-stable Lévy process with $\alpha \in (0, 2]$.

EM Scheme of SDE

The standard Euler-Maruyama (EM) scheme is given by:
$Y_0 = X_0 = x_0$ and

$$Y_{k+1} = Y_k + \eta f(Y_k) + g(Y_k)(L_{(k+1)\eta} - L_{k\eta}), \quad k \in \mathbb{Z}_+$$

where $\eta \in (0, 1)$ is the step size.

Known Result: the Brownian motion case

- When f is Lipschitz and g is bounded Lipschitz, EM scheme in a finite time interval $[0, T]$ strongly converges to SDE (0.1).
- A classical example is

$$\mathrm{d}X_t = -X_t^3 \,\mathrm{d}t + \mathrm{d}B_t, \quad X_0 = x_0 \in \mathbb{R}.$$

The corresponding EM scheme will blow up as the step size of EM scheme tends to zero.

The paper[1] considered the following assumption for f and g: There exist constants $\gamma > \lambda > 1$ and $H \geqslant 1$ such that for all $|x| \geqslant H$,

$$\max\{|f(x)|, |g(x)|\} \geqslant \frac{1}{H}|x|^{\gamma}, \text{ and, } \min\{|f(x)|, |g(x)|\} \leqslant H|x|^{\lambda}. \quad \textbf{(A)}$$

Then for any $p \in [1, \infty)$, the corresponding EM scheme blow up:

$$\lim_{N\to\infty} \mathbb{E}\left[\left|Y_N^N\right|^p\right] = \infty.$$

[1] Martin Hutzenthaler, Arnulf Jentzen, and Peter E. Kloeden. Strong and weak divergence in finite time of Euler's method for stochastic differential equations with non-globally Lipschitz continuous coefficients. Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., 467(2130):1563– 1576, 2011.

Examples for the blow up of EM scheme

(1) The **Ginzburg-Landau equation**:

$$\mathrm{d}X_t = \left(\left(a + \frac{1}{2}\sigma^2\right) X_t - bX_t^3\right) \mathrm{d}t + \sigma X_t \, \mathrm{d}B_t, \quad X_0 = x_0 \in (0, \infty),$$

for $t \in [0, T]$, where constants $a \geqslant 0$, $,\sigma > 0$. And the drift term satisfies

$$\left|\left(a + \frac{1}{2}\sigma^2\right) x - bx^3\right| \geqslant \frac{b}{2} |x|^3$$

for all $|x| \geqslant C \geqslant 1$.

(2) The **stochastic Verhulst equation**:

$$\mathrm{d}X_t = \left(\left(a + \frac{1}{2}\sigma^2\right) X_t - bX_t^2\right) \mathrm{d}t + \sigma X_t \, \mathrm{d}B_t, \quad X_0 = x_0 \in (0, \infty),$$

for $t \in [0, T]$.

Q1: Under Assumption (**A**), do we have the same result for α-stable noise with $\alpha \in (0, 2)$?

- **We prove that the EM scheme blows up**.

Q2: Let $f(x) = -x \log(1 + |x|)$, it is a critical case between $-x$ and $-x\,|x|^{\theta}$ with $\theta > 0$:

- $-x$: converge for Brownian motion and α-stable noise.
- $-x\,|x|^{\theta}$: blow up for Brownian motion and α-stable noise.
- $-x \log(1 + |x|)$: **what will happen for Brownian motion and α-stable noise?**

As $\alpha \in (0, 2)$, let $L_t = Z_t$ being a standard d-dimensional rotationally invariant α-stable process, and we consider the EM scheme

$$Y_{k+1} = Y_k + \eta\, f(Y_k) + g(Y_k)\left(Z_{(k+1)\eta} - Z_{k\eta}\right), \quad Y_0 = x_0,$$

where learning rate $\eta = T/n$ with $T > 0$ and $n \in \mathbb{N}$.

Theorem

[2] *We assume that* (**A**) *holds and* $g(x_0) \neq 0$ *for SDE* (0.1). *Let* $T > 0$ *be an arbitrary number and* $\eta = T/n$. *Then, for any* $\beta \in (0, \alpha)$, *we have*

$$\lim_{n\to\infty} \mathbb{E}\,|Y_n|^{\beta} = \infty$$

[2] X. Li, X., Y. Xiao: Phase transition in the EM scheme of an SDE driven by α-stable noises with $\alpha \in (0, 2)$, arXiv:2403.18626

SDE with drift $-x\log(1+|x|)$

Here, we consider the following SDE

$$\mathrm{d}X_t = -X_t \log\left(1+|X_t|\right)\mathrm{d}t + \mathrm{d}L_t, \quad X_0 = x_0 \in \mathbb{R}^d,$$

and corresponding EM scheme is

$$Y_{k+1} = Y_k - \eta Y_k \log\left(1+|Y_k|\right) + \left(L_{(k+1)\eta} - L_{k\eta}\right), \quad k \in \mathbb{Z}_+,$$

where $Y_0 = x_0$ and η is the learning rate.

- As $\alpha = 2$, denote $L_t = B_t$.
- As $\alpha \in (0,2)$, denote $L_t = Z_t$.

$\alpha = 2$, bounded in L^2

As $\alpha = 2$, the EM scheme is

$$Y_{k+1} = Y_k - \eta Y_k \log\left(1 + |Y_k|\right) + \left(B_{(k+1)\eta} - B_{k\eta}\right), \quad Y_0 = x_0.$$

We have that

Theorem

[3] *For any fixed initial value x_0, there exist constants $\eta_0 \leqslant \min\left\{(1+|x_0|)^{-2}, \mathrm{e}^{-5}\right\}$ and $C > 0$ such that for all $\eta \in (0, \eta_0]$,*

$$\sup_{m \geqslant 0} \mathbb{E}\,|Y_m|^2 \leqslant C.$$

[3] X. Li, X., Y. Xiao: Phase transition in the EM scheme of an SDE driven by α-stable noises with $\alpha \in (0, 2)$, arXiv:2403.18626

$\alpha \in (0, 2)$, Blow up

As $\alpha \in (0, 2)$, the EM scheme is

$$Y_{k+1} = Y_k - \eta Y_k \log(1 + |Y_k|) + \left(Z_{(k+1)\eta} - Z_{k\eta}\right), \quad Y_0 = x_0.$$

Theorem

[4] *Let $\alpha \in (0, 2), \beta \in (0, \alpha), T \in (0, \infty)$ be constants and let $\eta = T/n$ be the step size. For any $\beta \in (0, \alpha)$, we can find a T_β so that as $T > T_\beta$*

$$\lim_{n\to\infty} \mathbb{E}|Y_n|^\beta = \infty.$$

As $\alpha \uparrow 2$, the EM scheme demonstrates a phase transition.

[4] X. Li, X., Y. Xiao: Phase transition in the EM scheme of an SDE driven by α-stable noises with $\alpha \in (0, 2)$, arXiv:2403.18626

One page for the methods of the proofs

- $-x|x|^{\theta}$: we borrow the idea from the paper [5], the key point is to construct a special events so that the EM scheme will blow up on this event as the step size tends to infinity.
- $-x\log(1+|x|)+\alpha$-stable noise: the same as the above.
- $-x\log(1+|x|)$+Brownian motion: we split the Brownian motion into six regimes and use the strategy of 'split and conquer'.

[5] Martin Hutzenthaler, Arnulf Jentzen, and Peter E. Kloeden. Strong and weak divergence in finite time of Euler's method for stochastic differential equations with non-globally Lipschitz continuous coefficients. Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., 467(2130):1563– 1576, 2011.

For EM scheme

$$Y_{k+1} = Y_k - \eta Y_k \log\left(1 + |Y_k|\right) + \left(B_{(k+1)\eta} - B_{k\eta}\right), \quad Y_0 = x_0,$$

we $T = 100$, $n = 10000$ and $\eta = T/n = 0.01$, and consider three distinct initial points $Y_0 = 1$, 5 and 10 respectively. Then,

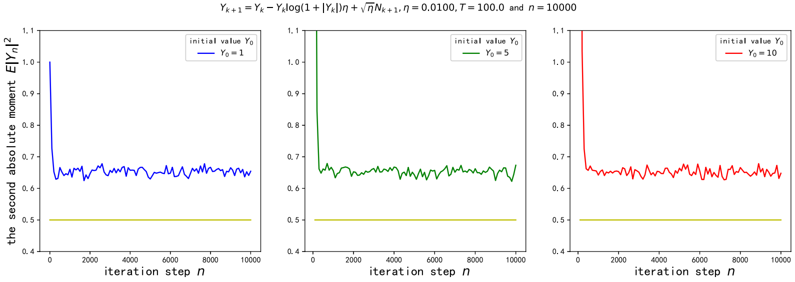

Figure 1: As $T = 100$, simulations values of $\mathbb{E}|Y_k|^2$ with initial $Y_0 = 1, 5, 10$, $\eta = 0.01$ and iteration steps $n = 10\,000$, $0 \leqslant k \leqslant n$.

Continue ...

In practice, $p_\alpha(t,x)$ does not have an explicit expression. Hence, the numerical simulation becomes complicated and computationally expensive. We can replace the stable noise $Z_{(k+1)\eta} - Z_{k\eta}$ with i.i.d. random variables with the Pareto distribution. That is,

$$\widetilde{Y}_{k+1} = \widetilde{Y}_k - \eta \widetilde{Y}_k \log\left(1 + \left|\widetilde{Y}_k\right|\right) + \frac{1}{\sigma}\eta^{1/\alpha}\widetilde{Z}_{k+1}, \quad \widetilde{Y}_0 := x_0, \quad (0.2)$$

for all $k = 0, 1, \ldots, n-1$. $\left\{\widetilde{Z}_k, k = 1, 2, \ldots\right\}$ is a sequence of i.i.d. Pareto-distributed random variables. we choose $T = 100$ here, and consider three cases, i.e., $\alpha = 0.5$, 1.0 and 1.5. For each case, we let β be $\alpha/8$, $\alpha/4$ and $\alpha/2$. Then, we can obtain the following tables.

Continue ...

n	$\mathbb{E}\lvert\widetilde{Y}_n\rvert^{\alpha/8}$	$\mathbb{E}\lvert\widetilde{Y}_n\rvert^{\alpha/4}$	$\mathbb{E}\lvert\widetilde{Y}_n\rvert^{\alpha/2}$
100	1.8×10^{13}	6.2×10^{26}	3.8×10^{55}
105	6.5×10^{13}	1.3×10^{28}	4.7×10^{57}
110	2.8×10^{14}	3.0×10^{29}	3.6×10^{60}
115	1.2×10^{15}	5.4×10^{30}	1.6×10^{63}
120	5.5×10^{15}	1.3×10^{32}	4.3×10^{65}
125	2.4×10^{16}	2.1×10^{33}	1.7×10^{68}
130	1.1×10^{17}	4.5×10^{34}	1.3×10^{71}
135	1.5×10^{18}	∞	∞
140	∞	∞	∞
145	∞	∞	∞

Table 1: Simulated values of the absolute moment for the EM scheme (**??**) with $T = 100$, $\alpha = 0.50$ and $n = \{100, 105, 110, \ldots, 145\}$.

Continue ...

n	$\mathbb{E}\vert\widetilde{Y}_n\vert^{\alpha/8}$	$\mathbb{E}\vert\widetilde{Y}_n\vert^{\alpha/4}$	$\mathbb{E}\vert\widetilde{Y}_n\vert^{\alpha/2}$
100	3.8×10^{25}	7.1×10^{52}	2.2×10^{109}
105	7.7×10^{26}	1.5×10^{55}	2.3×10^{112}
110	1.3×10^{28}	1.1×10^{58}	$5, 4 \times 10^{117}$
115	2.7×10^{29}	2.2×10^{60}	2.7×10^{124}
120	4.6×10^{30}	1.3×10^{63}	6.3×10^{128}
125	9.2×10^{31}	3.9×10^{65}	1.2×10^{135}
130	1.7×10^{33}	2.9×10^{68}	2.1×10^{141}
135	2.9×10^{34}	3.7×10^{71}	1.9×10^{145}
140	5.9×10^{35}	1.8×10^{73}	∞
145	∞	∞	∞

Table 2: Simulated values of the absolute moment for the EM scheme (**??**) with $T = 100$, $\alpha = 1.0$ and $n = \{100, 105, 110, \ldots, 145\}$.

Continue ...

n	$\mathbb{E}\lvert\widetilde{Y}_n\rvert^{\alpha/8}$	$\mathbb{E}\lvert\widetilde{Y}_n\rvert^{\alpha/4}$	$\mathbb{E}\lvert\widetilde{Y}_n\rvert^{\alpha/2}$
100	8.7×10^{37}	8.4×10^{77}	7.0×10^{158}
105	5.8×10^{39}	5.6×10^{83}	7.8×10^{168}
110	3.9×10^{41}	3.7×10^{86}	6.9×10^{174}
115	2.7×10^{43}	1.4×10^{90}	2.1×10^{182}
120	2.9×10^{45}	6.0×10^{93}	4.6×10^{192}
125	1.9×10^{47}	2.4×10^{97}	1.8×10^{200}
130	3.3×10^{49}	4.8×10^{101}	8.7×10^{207}
135	4.7×10^{51}	7.2×10^{104}	7.1×10^{215}
140	2.9×10^{53}	2.5×10^{111}	9.2×10^{219}
145	∞	∞	∞

Table 3: Simulated values of the absolute moment for the EM scheme (**??**) with $T = 100$, $\alpha = 1.5$ and $n = \{100, 105, 110, \ldots, 145\}$.

Thanks!